# Functorial Models of Differential Linear Logic


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## 1 Introduction and Motivation

The categorical formulation of differential settings was triggered by the study of denotational models of Differential Linear Logic[ER06] (DiLL), resulting in a variety of interdependent categorical definitions [BCLS20]. Differentiation in these settings might be axiomatized as an external operator acting on functions [BCS09], or as a hard-coded natural transformation refining models of linear logic [Fio07].

While studying models of Differential Linear Logic in functional analysis, polarities and their categorical models are most relevant. Indeed, having an involutive linear negation on topological vector spaces is often too much of a constraint, while having to duality leading to a contravariant equivalence of categories reinterprets a lot of already existing topological settings [Ker19]. Chiralities [Mel17] are a categorical axiomatization of polarized multiplicative linear logic, discovered by Melliès after a study of game models of linear logic. **In this abstract, we argue that while chiralities model the interaction between positive and negative formulas in linear logic, a similar structure models also the interaction between linear and non-linear proofs in** DiLL**.**

We see two advatantages to such a reformulation of models of DiLL: It is much closer to the models and the intuition that one might have in terms of differentiation. This makes proving that a concrete mathematical object is a model of DiLL much easier. It is categorically "pleasant" and easier to generalize to the case of Chiralities, which we will eventually want to do in order to express categorically polarized models of DiLL.

**Definition 1** ([Mel17])**.** A mixed chirality consists in two symmetric monoidal categories $(\mathscr{P}, \otimes, 1)$ and $(\mathscr{N}, \parr, \bot)$, between which there are two adjunctions, one of which being strong monoidal:

$$\begin{tikzcd} (\mathscr{P}, \otimes, 1) \arrow[r, bend left, "(-)^{\perp_P}"] \arrow[r, phantom, "\perp"] & (\mathscr{N}^{op}, \parr, \bot) \arrow[l, bend left, "(-)^{\perp_N}"] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{P} \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] & \mathscr{N} \arrow[l, bend left, "\downarrow"] \end{tikzcd} \tag{1}$$

with a family of natural bijections accounting for the monoidal closedness:

$$\chi_{p,n,m} : \mathscr{N}(\uparrow p, n \parr m) \sim \mathscr{N}(\uparrow(p \otimes n^{\perp_N}), m) \tag{2}$$

The natural bijections $\chi$ account for the lost monoidal closedness. They must respect the various associativity morphisms that we do not detail here. The chirality is said to be a *dialogue chirality* when the two adjunctions are equivalences. They are said to be negative chirality when the adjunctions are reflexive, positive when the adjunctions are co-reflexive.

Chiralities are in particular a model of polarized multplicative linear logic. They provide in particular the right setting for Nuclear Fréchet or DF spaces, providing the grounds to interpret distribution theory, and they allow for a nice reformulation of Banach-Steinhauss theorem [Ker19]. The central intuition to our work is that while the strong monoidal adjunctions of chiralities will model the usual linear non-linear adjunction of models of DiLL, the second adjunction will modelize the Differentiation, which is involutive on linear maps. Differentiation is not often modelized as a functor, and we will use the fact that Differentiation is functorial on the co-slice of the co-Kleisli. This is apparently a known fact on Cartesian Differential Categories.

## 2 Prerequisites

**Definition 2.** A **differential category**, called a 'differential storage category' in [**?** ], is a monoidal closed category $(\mathscr{L}, \otimes, 1)$ which is equipped with a biproduct $(\diamond, 0)$, a comonad $(!, d, \mu, d)$ such that $!$ is a strong monoidal functor $(\mathscr{L}, \diamond, 0) \longrightarrow (\mathscr{L}, \otimes, 1)$, and a natural transformation $\bar{d} : Id \longrightarrow !$ such that

$$\bar{d}_A; d_A = id_A \tag{3}$$

$$id_{!A} \otimes \bar{d}; \bar{c}; \mu = c \otimes \bar{d}; 1 \otimes \bar{c}; \mu \otimes \bar{d}_!; \bar{c}. \tag{4}$$

This makes in particular $!$ a lax monoidal endofunctor from $(\mathscr{L}, \otimes, 1)$ to $(\mathscr{L}, \otimes, 1)$ [Fio07] with monoidal law denoted $m$. The previous strong monoidal structure on $!$ induces a hopf algebra structure on each object $!A$, with natural transformation $w, c, \bar{w}, \bar{c})$. The requirement on $\bar{d}$ translate the fact that the differential of linear maps is the identity, and validate the chain rule, meaning that for morphisms of the co-Kleisli $\mathscr{L}_!$, for any point $a$ $D_a(g \circ f) = D_{f(a)} g \circ D_a f$

Differential categories are denotational models of intuitionistic DiLL, and of classical DiLL when they are $*$-autonomous. We want to adapt this definition to the linear/non-linear adjunction, meaning that instead of a comonad $!$ we will consider a strong monoidal adjunction

$$\begin{tikzcd} (\mathscr{C}, \times, I) \arrow[r, bend left, "\mathcal{E}'"] \arrow[r, phantom, "\perp"] & (\mathscr{L}, \otimes, 1) \arrow[l, bend left, "\mathcal{U}"] \end{tikzcd}; \tag{5}$$

such that $! := \mathcal{E}' \circ \mathcal{U}$, and $\mathscr{L}$ is equipped with a biproduct.

**Definition 3.** Let $I$ be an object of a category $\mathscr{C}$. We recall that the *co-slice* category $(I \downarrow \mathscr{C})$ has as objects arrows $a : I \longrightarrow A$ in $\mathscr{C}$ and as morphisms $f : (a : I \longrightarrow A) \longrightarrow (b : I \longrightarrow B)$ those morphisms $f : A \longrightarrow B$ of $\mathscr{C}$ such that $a; f = b$. If $f : A \longrightarrow B$ is an arrow of $\mathcal{C}$, and $a$ an object in $(I \downarrow \mathscr{C})$ we denote by $(a|f)$ the morphism $(I \downarrow \mathscr{C})$ from $a$ to $a; f$ induced by $f$.

The key idea to our work is that within a linear non-linear ajdunction, the functoriality $\overrightarrow{\mathcal{D}} : (I \downarrow \mathscr{C}) \longrightarrow \mathscr{L}$ expresses the chain rule (4). One would think that asking $\overrightarrow{\mathcal{D}}$ to be the left inverse to the canonical extention of $\mathcal{U}$ to $(I \downarrow \mathscr{C})$ would be enough:

$$\bar{U} : \ell \mapsto (0|\ell) \qquad \overrightarrow{\mathcal{D}} \circ \bar{U} = Id_{\mathscr{L}}.$$

However, as far as we can see, one needs to enforce the involutivity of $\overrightarrow{\mathcal{D}}$ on every linear map $\mathcal{U}(\ell)$ at every point $a$ to land back on a differential category. This is expressed through the following

**Definition 4.** Given an object $I$ of a category $\mathscr{C}$, and a functor $\mathcal{U} : \mathscr{L} \longrightarrow \mathscr{C}$, we recall the notion of Category of Generalized Elements of $\mathcal{U}$ (over $I$): $El_I(\mathcal{U})$, which has as objects the arrows $a : I \longrightarrow \mathcal{U}(A)$ in $\mathscr{C}$ and as morphisms from $(a : I \longrightarrow A)$ to $(b : I \longrightarrow B)$ those morphisms $\mathcal{U}(f) : \mathcal{U}A \longrightarrow \mathcal{U}B$ of $\mathscr{C}$ such that $a; \mathcal{U}f = b$. This categorically corresponds to taking the pullback of $\mathcal{U}$ and the projection of $\Pi_{\mathscr{C}} : (I \downarrow \mathscr{C}) \longrightarrow \mathscr{C}$

$$\begin{tikzcd} (El_I(\mathcal{U}), \times, I) \arrow[r, "\overrightarrow{\mathcal{U}}"] \arrow[d, "\Pi_{\mathscr{L}}"'] & ((I \downarrow \mathscr{C}), \times, I) \arrow[d, "\Pi_{\mathscr{C}}"] \\ (\mathscr{L}, \otimes, 1) \arrow[r, "\mathcal{U}"'] & (\mathscr{C}, \times, I) \end{tikzcd} \tag{6}$$

# 3 Functorial Models of DiLL

We now can define an equivalent structure to differential storage categories:

**Definition 5.** A functorial model of intuitionistic DiLL consists of the following :

- A monoidal closed category $(\mathscr{L}, \otimes, 1)$, admitting a biproduct $\diamond$.
- A cartesian monoidal category $(\mathscr{C}, \times, I)$. Since the forgetful functor $(I \downarrow \mathscr{C}) \longrightarrow \mathscr{C}$ creates limits, the coslice inherits finite products from $\mathscr{C}$.
- A **linear–non-linear adjunction** between $\mathscr{L}$ and $\mathscr{C}$, which is a strong monoidal adjunction (Diagram 6).
- A functor $\overrightarrow{\mathcal{D}} : (I \downarrow \mathscr{C}) \longrightarrow \mathscr{L}$ such that the following diagram commutes:

$$\begin{tikzcd} (El_I(\mathcal{U}), \times, I) \arrow[r, "\vec{\mathcal{U}}"] \arrow[d, "\Pi_{\mathscr{L}}"'] & ((I \downarrow \mathscr{C}), \times, I) \arrow[d, "\Pi_{\mathscr{C}}"] \arrow[dl, "\overrightarrow{\mathcal{D}}"] \\ (\mathscr{L}, \otimes, 1) \arrow[r, "\mathcal{U}"'] & (\mathscr{C}, \times, I) \end{tikzcd} \tag{7}$$

  This implies in particular that for any object $A$ of $\mathscr{L}$ we have $\overrightarrow{\mathcal{D}}(\mathcal{U}(A)) = A$, $\overrightarrow{\mathcal{D}}(a|\mathcal{U}(\ell)) = \ell$ and that $\overrightarrow{\mathcal{D}}$ preserves product.

- We also ask that $\mathscr{L}$ is well pointed with respect to $1$.

Intuitively, the functoriality of $\overrightarrow{\mathcal{D}}$ encodes the chain rule, and the commutation of the diagram the fact that the differential of a linear function is itself. Let us denote $u_A : 0 \longrightarrow A$ the inital morphism in $\mathscr{L}$, and $\chi_{A,B} : \mathscr{C}(A, \mathcal{U}(B)) \simeq \mathscr{L}(\mathcal{E}'(A), B)$ the natural isomorphisms resulting from the adjunction 6. We define the natural transformation interpreting the co-dereliction rule of DiLL as follows:

$$\bar{d}_A := \overrightarrow{\mathcal{D}}\left(\mathcal{U}(u_A)|\chi^{-1}(id_{!A})\right) : A \multimap !A.$$

This leads to the following equation expressing the differential at any point, and that leads, after a bit of work, to the interpretation of equation (4).

$$\overrightarrow{\mathcal{D}}(a|\chi^{-1}(\ell)) = \bar{d}_A \otimes \mathcal{E}'(a); \bar{c}; \ell$$

As $u_A$ is the unique morphism with the required domain and codomain, and by functoriality of $\overrightarrow{\mathcal{D}}$, we have the following important fact:

**Lemma 6.** Let $a : I \longrightarrow \mathcal{U}(A)$, $f : \mathcal{U}(A) \longrightarrow B$, $g : B \longrightarrow C$ be morphisms in $\mathscr{C}$. Then we have:

$$\overrightarrow{\mathcal{D}}(\mathrm{id}_I\,|a) = u_A \text{ and } \overrightarrow{\mathcal{D}}(a|f; g) = \overrightarrow{\mathcal{D}}(a|f); \overrightarrow{\mathcal{D}}(a; f|g)$$

Because $\overrightarrow{\mathcal{D}}$ preserves products, and interprets morphisms $(a|\mathcal{U}(\ell))$ by $\ell$, we have the following key lemma, which interprets the action of $\overrightarrow{\mathcal{D}}$ at different points of a function $g$.

**Lemma 7.** The functor $\overrightarrow{\mathcal{D}}$ can internalize translations, meaning for any object $a : I \longrightarrow \mathcal{U}(A)$ and morphisms $(a|f)$ and $(a; f|g)$ in $(I \downarrow \mathscr{C})$, we have:

$$\overrightarrow{\mathcal{D}}(a|g) = \overrightarrow{\mathcal{D}}(\mathcal{U}(u_A)|(id_{\mathcal{U}(A)} \times a); \mathcal{U}(\nabla); g).$$

The two next lemmas prove that our transformation is indeed the differentiation of differential categories, first by doing it in a context-free environment and then generalizing it to the commutation of $\bar{d}$ with promotion within a context.

**Lemma 8.** The co-dereliction $\bar{d}_A$ agrees with the diagram reflecting the chain rule in differential categories according to [Fio07], which corresponds to the alternate chain rule $[dC.4']$ in [BCLS20]:

$$\bar{d}; \mu = \bar{d} \otimes !(u_A); d_! \otimes \mu; \bar{c}$$

**Lemma 9.** The co-dereliction $\bar{d}_A$ agrees with the generalized chain rule in differential categories (the diagram $[dC.4]$ in the survey [BCLS20]):

$$id_{!A} \otimes \bar{d}; \bar{c}; \mu = c \otimes \bar{d}; 1 \otimes \bar{c}; \mu \otimes \bar{d}_!; \bar{c}.$$

**Theorem 10.** *A functorial model of Intuitionnistic* DiLL *is a differential storage category. A well-pointed differential storage category is a functorial model of Intuitionnistic* DiLL.

# 4 Generalization of Chiralities

For the definition of the previous section to be an appropriate generalization of chiralities, we would need to consider a pair of a covariant and a contravariant adjunction. The contravariant strong monoidal adjunction is retreived from the usual linear/non-linear adjunction by considering $*$-autonomous categories. One defines a *functorial model of classical* DiLL and consider a $*$-autonomous categorie $\mathscr{L}$ with the following structure:

$$\begin{tikzcd} (\mathscr{C}, \times, I) \arrow[r, bend left, "\mathcal{E}"] \arrow[r, phantom, "\perp"] & (\mathscr{L}^{op}, \otimes, 1) \arrow[l, bend left, "\mathcal{U}'"] \end{tikzcd}; \qquad \begin{tikzcd} (El_I(\mathcal{U}), \times, I) \arrow[r, "\overrightarrow{\mathcal{U}}"] \arrow[d, "\Pi_{\mathscr{L}}"'] & ((I \downarrow \mathscr{C}), \times, I) \arrow[d, "\Pi_{\mathscr{C}}"] \arrow[dl, "\overrightarrow{\mathcal{D}}"] \\ (\mathscr{L}, \otimes, 1) \arrow[r, "\mathcal{U}"'] & (\mathscr{C}, \times, I) \end{tikzcd} \tag{8}$$

where $\mathcal{U}'$ is the composition of $\mathcal{U}$ with the intepretation of $(\_)^{\perp}$, and $\mathcal{E}$ is thought of the contravariant hom-set of non-linear scalar maps $\mathcal{E}(A) = \mathcal{C}(A, \mathbb{K})$.

We now show how our functorial axiomatization of models of classical DiLL restrict to chiralities. Indeed, we did not ask for two adjunctions, but for a strong monoidal adjunction and a commuting diagram involving a category of elements. We will here look at the chirality setting, in the particular case where it refines a $*$-automous additive category (as is the case is models of classical DiLL). Consider a pair of categories both with an initial object, and a functor $\mathcal{F}$ preserving colimits (typically a left adjoint):

$$(\mathscr{P}, 0_{\mathscr{P}}) \xrightarrow{\mathcal{F}} (\mathscr{N}, 0_{\mathscr{N}})$$

Then one can generalize $\mathcal{F}$ to a functor between the category of elements $El'_{0_{\mathscr{N}}}(\mathcal{F})$ and $\mathscr{N} \downarrow 0_{\mathscr{N}}$. Asking for the existence of a functor $\overrightarrow{\mathcal{G}}$ commuting as below mimicks the situation of the previous section, where $\mathscr{N}$ was $\mathcal{C}$ and $\mathscr{P}$ was $\mathscr{L}$:

$$\begin{tikzcd} (\mathscr{P}, \otimes, 1) \arrow[r, bend left, "(-)^{\perp_P}"] \arrow[r, phantom, "\perp"] & (\mathscr{N}^{op}, \parr, \perp) \arrow[l, bend left, "(-)^{\perp_N}"] \end{tikzcd} \qquad \begin{tikzcd} El'_{0_{\mathscr{N}}}(\mathcal{F}) \arrow[r, "\overrightarrow{\mathcal{F}}"] \arrow[d, "\Pi_{\mathscr{P}}"'] & (\mathscr{N} \downarrow 0_{\mathscr{N}}) \arrow[d, "\Pi_{\mathscr{N}}"] \arrow[dl, "\overrightarrow{\mathcal{G}}"] \\ (\mathscr{P}, 0_{\mathscr{P}}) \arrow[r, "\mathcal{F}"'] & (\mathscr{N}, 0_{\mathscr{N}}) \end{tikzcd} \tag{9}$$

This diagram also states (for free) a weak statement of adjunction: $El'_{0_{\mathscr{N}}}(F)(N_1, N_2) \simeq \mathscr{N}(N_1, 0) \times \mathscr{P}(N_1, N_2)$. Let us show that this generalizes the right adjunction of chiralities (Diagrams 1). Indeed, in that particular case we have that 0 is terminal in $\mathscr{N}$, and as such $\mathscr{N} \downarrow 0_{\mathscr{N}} \equiv \mathscr{N}$, $\Pi_{\mathscr{N}} = Id$. Diagram 9 then simplifies as follows:

$$\begin{tikzcd} El'_{0_{\mathscr{N}}}(\uparrow) \arrow[r, "\overrightarrow{\uparrow}"] \arrow[d, "\Pi_{\mathscr{P}}"'] & (0_{\mathscr{N}} \downarrow \mathscr{N}) \arrow[d, "\Pi_{\mathscr{N}}"] \arrow[dl, "\downarrow"] \\ (\mathscr{P}, 0_{\mathscr{P}}) \arrow[r, "\uparrow"'] & (\mathscr{N}, 0_{\mathscr{N}}) \end{tikzcd} \rightsquigarrow \begin{tikzcd} & El'_{0_{\mathscr{N}}}(\uparrow) \arrow[dl, "\Pi_{\mathscr{P}}"'] \arrow[dr, "\overrightarrow{\uparrow}"] & \\ (\mathscr{P}, 0_{\mathscr{P}}) \arrow[rr, bend right, "\uparrow"'] & & (\mathscr{N}, 0_{\mathscr{N}}) \arrow[ll, "\downarrow"'] \end{tikzcd} \tag{10}$$

Be careful of the poor notational coincidence: the functor $\downarrow$ has nothing to do with $\mathcal{C} \downarrow 0$. The commutation of the below part of the diagram below expresses the fact that $\downarrow\uparrow = Id$. In diagram 9, one could express the adjunction between $\mathcal{E}$ and $\mathcal{U}'$ as relative $\mathcal{E}' - \Pi_{\mathcal{C}}$ ajdunction between $\mathcal{U}$ and $\overrightarrow{\mathcal{D}}$:

$$\mathscr{L}((\mathcal{U}' \circ \mathcal{E})(A), \overrightarrow{\mathcal{D}}(b, B)) \simeq \mathscr{C}(\mathcal{U}(A), \Pi_{\mathcal{C}}(b, B)) \quad \Leftrightarrow \quad (\mathscr{L}(!A, B) \simeq \mathscr{C}(A, B)) \tag{11}$$

*Requiring* this relative adjunction in diagram 9 gives us the adjunction between $\uparrow$ and $\downarrow$, as $()^{\perp_N \perp_P} \simeq Id_{\mathscr{N}}$ and $\Pi_{\mathscr{N}} \simeq Id$. Therefore, adding equation 11 to diagram 8, **we have defined a new structure, which generalizes chiralities and axiomatizes functorially models of classical** DiLL.

# 5 Future work

This is work in progress and there is still much to explore. As a fun fact, let's notice that we $\mathscr{L}$ is a calculus category [CL18], that is a category with both differentiation and integration, then we have an relative $! \otimes Id$-adjunction between $(I \downarrow \mathscr{C})$ and $\mathscr{L}$, where the fundamental theorem of analysis is expressed exactly as a relative $! \otimes Id$ adjunction between $\overrightarrow{\mathcal{D}}$ and an extension of U:

$$\begin{tikzcd} (I \downarrow \mathscr{C}) \arrow[rr, bend left, "\overrightarrow{\mathcal{D}}'"] \arrow[rr, phantom, "\perp"] & & \mathscr{L} \arrow[ll, bend left, "\bar{\mathcal{U}}"] \end{tikzcd} \qquad (I \downarrow \mathscr{C})((a, A), B) \simeq \mathscr{L}(!A \otimes A, B) \qquad \bar{\mathcal{U}}(A) = \mathcal{U}(u_A)$$

Once this work is stabilized, we would like to express more precisely what polarized models of DiLL should be, and refine diagram 8 when $\mathscr{L}$ is not star-automous but decomposed in a polarized chirality as in diagram 1.

While our setting expresses the chain rule, it's also fun to see what happens in the symmetric setting and how it express the "co-chain" rule of exponential maps [KL23]. In that case, ! is a monad and the object of the slice on the Kleisli category are morphisms $a' : A \multimap \mathbb{K}$, that is elements of the dual of $A$. The co-chain rule is expressed in a generalized exponential map: $E : \mathscr{L}^! \downarrow I \longrightarrow \mathscr{L}$ Finally, this setting gives a dependent flavor to differentiation, and we would like to investigate a possible link with dependent types.

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